Final Exam - Analysis IV
B. Math III

24 April, 2023

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 110).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

For $f \in L^1[-\pi, \pi]$, the Fourier coefficients of f are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos nx\ dx, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\ dx.$$

The series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the *Fourier series* of f. The nth Cesaro mean of the Fourier series for f is denoted by $\sigma_n(f)$.

1. Let $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, and let $\mathscr{A}$ be the algebra of functions on S^1 of the form,

$$f(z) = \sum_{n=0}^{N} c_n z^n,$$

where $c_n \in \mathbb{C}$.

(a) (10 points) Show that $\mathscr{A}$ separates points on S^1 and vanishes at no point of S^1.

(b) (5 points) Show that there are continuous complex-valued functions on S^1 which are not in the uniform closure of $\mathscr{A}$.

Total for Question 1: 15

2. Let $\{f_n\}$ be a sequence of bounded real-valued functions on $[0,1]$, and let F_n be defined by

$$F_n(x) := \int_0^x f(t)\,dt.$$

(a) (5 points) Show that the sequence $\{F_n\}$ has a convergent subsequence if there is some $M > 0$ such that $\|f_n\|_\infty < M$ for all $n \in \mathbb{N}$.

(b) (10 points) Show that the sequence $\{F_n\}$ has a convergent subsequence if there is some $M > 0$ such that $\int_0^1 |f_n(t)|^2\,dt < M$ for all $n \in \mathbb{N}$.

Total for Question 2: 15

3. (20 points) Suppose that f is a 2π-periodic function on $\mathbb{R}$ that satisfies the Lipschitz condition of order α ($0 < \alpha \leq 1$); that is $|f(x+h)-f(x)| \leq C|h|^\alpha$ for $C > 0$ independent of x. Show that if a_n, b_n are Fourier coefficients of f, then

$$a_n = O(n^{-\alpha}), b_n = O(n^{-\alpha}).$$

Total for Question 3: 20

4. (20 points) Assume that f is a 2π-periodic function integrable on $[-\pi,\pi]$ and that f is of bounded variation on $[x_0-\delta, x_0+\delta]$. Show that the Fourier series of f at x_0 converges to $\frac{1}{2}(f(x_0^+) + f(x_0^-))$.

Total for Question 4: 20

5. (20 points) Assume that f is 2π-periodic on $\mathbb{R}$ and $f \in L^1[-\pi,\pi]$. Then show that

$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_1 = \lim_{n\to\infty} \int_{-\pi}^{\pi} |\sigma_n(f)(x) - f(x)|\,dx = 0.$$

Total for Question 5: 20

6. (20 points) Show that

$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Total for Question 6: 20